K-SAT reduces to VERTEX-COVER

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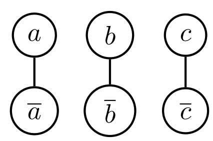
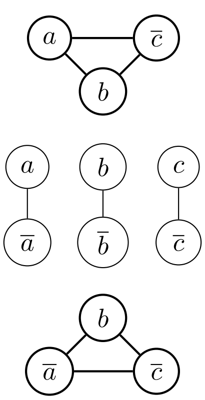
Theorem. *For every formula ϕ in K-cnf with C clauses and V variables, we can construct a graph G in polynomial time such that ϕ is satisfiable iff G has a k-vertex cover, where $k = C(K-1)+V$.*

Proof. We begin by constructing G using two gadgets. For all V variables, we construct a *variable gadget*, which consists of the symbols x and $\overline{x}$ for some variable x. For example, if ϕ uses the variables a, b, c then G starts as the following:



This takes $\mathcal{O}(V)$ time, since we construct $2V$ nodes and V edges.

Next, we construct C *clause gadgets* (one per clause). These are K-cliques that contain the variables in a certain clause. Continuing the previous example, if $C = 2$, $K = 3$, and $\phi = (a \vee b \vee \overline{c}) \wedge (\overline{a} \vee b \vee \overline{c})$ then we would add two 3-cliques to G:

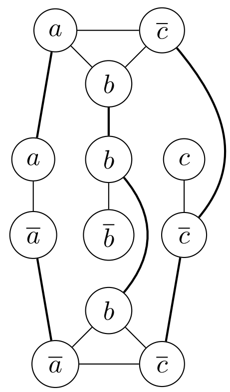


We will call each of these K-cliques a *clause gadget*. Each of the C clauses takes

$$\mathcal{O}\left(K + \binom{K}{2}\right) = \mathcal{O}\left(K^2\right)$$

time to construct, so it takes $\mathcal{O}\left(CK^2\right)$ time to construct all of them.

Finally, we connect each of the CK variables to their respective variable gadgets:



This takes $\mathcal{O}(CK)$ time. In total, constructing G takes

$$\mathcal{O}(V) + \mathcal{O}\left(CK^2\right) + \mathcal{O}(CK) = \mathcal{O}\left(V + CK^2\right)$$

time, which is clearly polynomial time since C, K, and V are bounded by the length of $\langle \phi \rangle$.

(A) *How do we know that ϕ is satisfiable if G has a k-vertex cover, where* $k = C(K-1) + V$? Let Ψ be a k-vertex cover of G.

(1) Every clause gadget contributes at least $K - 1$ vertices to Ψ since it is a connected graph with K nodes. In total, the C clause gadgets contribute at least $C(K-1)$ vertices to Ψ.

(2) Every variable gadget contributes at least 1 vertex to Ψ since it has an internal edge. In total, the V variable gadgets will contribute at least V vertices to Ψ.

Combining (1) and (2), we see that *exactly* $K-1$ nodes are included from every clause gadget and *exactly* 1 is included from each variable gadget—otherwise, we would have more than $C(K-1)+V$ nodes. Note that in each of the C clauses, the one node that was not included will have its symbol included in the corresponding variable gadget. Otherwise, the clause-variable edge will not be covered. Since the V symbols included from variable gadgets are non-conflicting (1 per gadget), we can construct the a satisfying set S which contains these V symbols. The set will satisfy ϕ because each clause has one node whose corresponding symbol was included in Ψ.

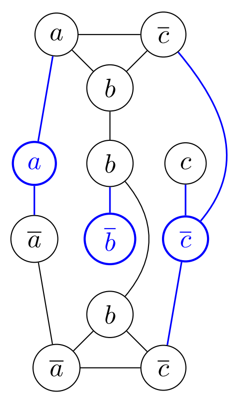
(B) *How do we know that G has a k-vertex cover if ϕ is satisfiable?* Given a satisfying set S of V symbols, we can construct a k-vertex cover for G by doing the following:

(1) For every symbol in S, include in the cover the corresponding symbol from the variable's gadget. (This includes V nodes in the cover.)

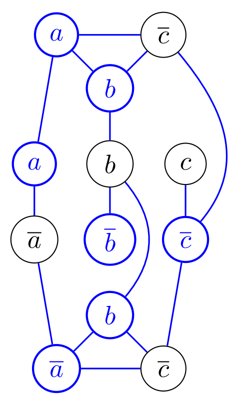
(2) For every K-clique that represents a clause, include $K-1$ of the nodes in the cover, leaving out a node that is also present in S. There will be at least one such node; otherwise S isn't a satisfying set. (This includes $C(K-1)$ nodes in the cover.)

These two steps create a k-vertex cover, where $k = C(K-1) + V$. This cover will cover all edges because all edges within the C clause gadgets will be covered by the $K-1$ nodes included from each gadget; also, since the one node that isn't included from the clause gadget is present in S, its symbol will be included in the corresponding variable gadget, which covers that clause-variable edge; finally, since S is a satisfying set, all variables $\star$ will be present in S (either as $\star$ or $\overline{\star}$), which means the edge within each of the V variable gadgets will be covered.

Continuing the previous example, notice that one possible S would be $\{a, \overline{b}, \overline{c}\}$ since $\phi = 1$ when $a = \overline{b} = \overline{c} = 1$. After (1), we would have the following nodes included:



After (2), we would indeed have a k-vertex cover, where $k = 2(3-1) + 3 = 7$:



Combining (A) and (B) we get that ϕ is satisfiable iff G has a k-vertex cover, where $k = C(K-1) + V$. □